Cryptocurrency Adoption: The Role of Speculative Price Bubbles in Product Diffusion

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Abstract

We study product adoption in the context of a currency market. Currencies are subject to network effects and speculative investments, which are not part of standard models of product diffusion. To explore this unique setting, we marry models of stochastic bubbles and the standard model of product diffusion. A rational bubble is raised due to speculative investors seeking short-term gains. The participation of investors helps establish the value of the currency (as a medium of exchange). We find that a bubble accelerates the adoption, which can help explain the fast diffusion of Bitcoin. There are reinforcing interactions between the speculative investors and regular users of currency, which can make it easier to form a bubble (compared to a setting without regular users). We also provide conditions under which bubbles may unravel.

Keywords: Product Adoption, Currency Marketing, Speculative Bubbles, Cryptocurrency

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1 Introduction

Currencies have been utilized for millennia as mediums of exchange in economic transactions. Governments have historically marketed new currencies to help facilitate trade in their economies. The most recent major currency launched by a government institution was the Euro in 1999. Recently, even non-government agents are able to market new currencies in digital formats. Since Bitcoin emerged in 2009, there have been more than 1,600 crypto-currencies introduced.\footnote{Even Facebook plans to launch a digital currency in the near future. ("Facebook’s planned new currency may be based on blockchain," The Economist, May 30, 2019.)} This research studies product diffusion in the context of a new currency.

A important feature of a currency is that it is particularly susceptible to speculative price bubbles, caused by investors who exploit currency exchange for speculative gains. Speculative beliefs of investors, very often detached from the fundamental value of a currency, can drastically affect the price that non-investors, the users, pay to utilize the currency as a medium of exchange. A second important feature of a currency is its network externality. A user’s benefit from adoption depends on the number of other users with whom to exchange. The symbiotic relationship between this network effect and speculative incentives implies adoption dynamics that are different than other new products (e.g. Bass 1969). In light of this relationship, we ask: How does the presence of investors, who are interested in speculative returns, affect the diffusion of a new currency?; and How does product adoption by users affect the creation of speculative price bubbles?

These questions are connected to the recent and heated attention around Bitcoin, especially in the years 2013 and 2017, which witnessed surges in Bitcoin price. It is tempting to attribute the price surges to the growth in the user base or optimistic expectation of such growth. But we also entertain the reverse mechanism by asking whether price bubbles may have contributed to growth of the Bitcoin usage. Indeed, absent a formalized model, it is hard to say whether the presence of investors helps or hurts product adoption by users. On one hand, if investors’ speculative incentives generate price bubbles, then users might be priced out of adoption. On the other hand, the attraction of a currency for speculative returns could boost adoption by virtue of
the network effects spurred by investors.

Our formalization starts by considering the demand for currency speculation by investors only. This benchmark illustrates conditions for which bubbles can sustain. Investors decide whether to participate in the currency market based on expectations about tomorrow’s price of the currency. Their expectations are rational, meaning that they are consistent with the realization of tomorrow’s price. Investors also rationally expect crashes – events where the price drops to zero. In other words, our analysis focuses on fully rational investors who seek out short-term gains. We refrain from adding behavioral factors (such as emotional responses or fears) to examine the most demanding environment for bubbles to form. Without behavioral factors, the key requirement in the formation of bubbles is investor confidence (Weil 1987). A product can be valued above its fundamental value only if there is a minimum degree of confidence among agents that the currency will be valuable in the future, which, if fails to happen, implies a crash.

Next, we introduce currency adoption by users who demand currency as a medium of exchange and have no desire for speculative returns. The simultaneous presence of users and investors implies a mutual interdependence on the equilibrium price path of the currency. Specifically, users must take into account the additional network externality implied by investors, who also accept it for transactions. Simultaneously, investors take into account users’ demand for currency as a medium of exchange as it affects the price tomorrow and subsequently the speculative return. Our model formally accounts for this interdependent demand and allows us to discern between the currency’s speculative value and its fundamental value and their complementary roles in its diffusion. The currency’s fundamental value is defined as its price when no speculative investors participate in the market. Investors can drive up the price beyond this value, generating speculative value, in the form of a price bubble. A bubble may burst – dropping the price to the fundamental value.

We establish two results. First, the adoption of a currency is accelerated by the bubble generated by speculative investors. The literature on product diffusion has focused on products without financial or investment value (Bass, 1969). Our model demonstrates that the price bubbles induce faster adoption rates and more adoption than without investors. This result has implications for any government or non-
government agent whose objective is to induce diffusion of a currency. Specifically, it suggests that opening up the currency to investors at its launch can help to achieve adoption goals. Indeed, it is worthwhile to note that the European Monetary Union initially launched the Euro in 1999 only to investors, before rolling it out as a physical currency in 2002.\footnote{See Wikipedia, https://en.wikipedia.org/wiki/Euro.}

Our theory provides a rationale for such a launch strategy. Our second result states that the presence of users relaxes the conditions for a bubble to form and grow. Specifically, we show that the level of investor confidence needed to sustain a bubble decreases when there is more fundamental demand from users. Intuitively, the relaxation comes from an increased expectation in the currency’s future fundamental value due to user adoption. What is interesting, however, is that even in the case where investing in fundamental value provides zero expected return,\footnote{This is modeled by introducing a technological or legal risk where the Bitcoin may cease to exist. Without such a risk, the return from investing in fundamental value will be always positive due to the fact that user adoption is monotonically increasing over time as in the Bass model.} we show that such relaxation still holds. This is due precisely to the reinforcing interplay between investors and users, which raises an investor’s expected return beyond the return from investing in fundamental value.

These effects may help us understand the path of Bitcoin prices, which is shown in Figure 1. Consider the steep ascent, around November 2013, when the Shared-Coin service was offered publicly for free. This service offered anonymity in transactions by mixing crypto-currency funds to obscure the trail back to a fund’s original source (Athey et al. 2016). Arguably, this new service reduced the cost of adoption by users. Through the lens of our model, the corresponding and subsequent volatility starting in late 2013 can be explained in a completely rational way by the relaxed confidence requirement to trigger a bubble. In other words, as Shared-Coin attracted more users, the necessary confidence for bubbles had been relaxed permitting easier conditions for speculative gain. By the end of 2015, the price stabilizes around $250, which is greater than the price before Shared-Coin’s introduction. This, according to our model, would be reflective of the increased fundamental value of Bitcoin as a medium of exchange.\footnote{Our model tells a similar story to explain another major price spike of Bitcoin in late 2017. Earlier in 2017, several economies, Japan, Russia, and Norway, announced to legitimize the crypto-currency, thereby lowering the adoption costs for users and subsequently, according to our model, relaxing the...}
The above story is not without limits, however. We find that a key moderator is the assumed level of impact each investor has on the currency market relative to an adopter. As long as this relative impact is neither too strong nor too weak, then a price bubble can exist. Intuitively, investors buying currency (i) drive up today’s price, and (ii) stimulate user adoption and thus increase the expectation of tomorrow’s price. The sustenance of bubble requires the balance of these two effects which essentially maintain the expected return (ratio of today’s and tomorrow’s price) at a reasonable level. An investor’s relative impact on the currency market affects the magnitude of (i) but does not directly affect (ii). Hence, when the investors’ relative impact is either too large or too small, a bubble becomes unsustainable for any level of investor confidence and, as early investors rationally anticipate this lack of sustainability, they refuse to buy the currency and the bubble unravels.

Our work combines the product diffusion literature in marketing with the currency formation and asset bubbles literature in macro-finance. In marketing, starting with Bass (1969), the product diffusion literature examines the path of consumer adoption following the introduction of an innovative product. Subsequent work augments the classic work of Bass (1969) to include imitators and influencers (Steffens and Murthy, confidence required for a bubble.
1992; Van den Bulte and Joshi 2007). Much of that prior work implicitly assumes network externalities where the probability that a user adopts depends on the latest adopter population. Network externalities are a key property of currency and are an essential aspect of our main effects. There have also been great interests in marketing on the empirical evaluation of diffusion models and network externalities (e.g., Bass et al., 1994; Garber et al., 2004; Bronnenberg and Mela, 2004; Shriver, 2015). However, neither the theory nor empirical literature has explored the role of price bubbles.

On the finance side, there are several ground-laying theory works on asset bubbles. Blanchard (1979) first points out that bubbles do not necessarily indicate irrational behaviors – speculative bubbles followed by market crashes can be consistent with the assumption of rational expectations. Tirole (1982, 1985) show that under rational expectation, short-term investors (instead of infinitely lived agents) are necessary for sustaining speculative detachment from fundamental values. As a result, our model features investors seeking short-term gains. Tirole only considered deterministic models. Weil (1987) extends the analysis to stochastic bubbles with the possibility of crashes. In doing so, he introduces the concept of “confidence:” a bubble can exist only if there is enough confidence that it will persist (not crash). Following his work, we focus on evaluating how the confidence level required for a bubble to exist is affected by the presence of user adoption.

Since then, there have been more studies exploring factors behind the formation of bubbles. While largely maintaining the assumption of rational expectations, these studies focus on various deviations from perfect markets, i.e., market frictions. These include asymmetric information (Allen and Gorton 1993), the inability of arbitrageurs to synchronize selling strategies (Abreu and Brunnermeier 2003), risk aversion (Branch and Evans 2011), and portfolio constraints (Hugonnier 2012). Our objective is not to provide another explanation of bubbles, per se, but rather to understand the interplay between speculative incentives and product adoption, or put differently and more broadly, the mutual impact between investors and users.
2 The Market for Currency

The demand for currency in our model comes from two sources: investors and users. Investors demand currency on the expectation that they can sell it in the future for a speculative gain. Users demand currency to conduct transactions with others. Users can be interpreted as merchants and customers exchanging goods and services. We start by examining the demand for currency by investors only. By doing so, we illustrate how the market for currency can survive from purely speculative incentives. We then incorporate user demand for the full model to capture how these two sources of currency demand interact. Later, in Section 3, we derive properties of the full model and compare them with the investor-only case.

2.1 Investors Only

There are $T$ discrete periods. In every period $t = 1, ..., T$, an investor in $t$ is endowed with $y$ dollars. She may invest in either the coins or an outside option. For simplicity, assume that the outside option provides an interest rate $r = 0$. Let $p_t$ denote the price of a coin in period $t$. An investor decides whether to buy coins in period $t$ based on the belief about the next period’s price, $\mathbb{E}(p_{t+1})$. We parametrize the belief in the following way,

$$p_{t+1} = \begin{cases} L\omega_t, & \text{with probability } \omega_t; \\ 0, & \text{with probability } 1 - \omega_t. \end{cases} \tag{1}$$

The investor’s belief, $\omega_t$, represents the probability a high price will sustain tomorrow and $L > 0$ is the maximum possible price level. Both $\omega_t$ and $L$ are determined by rational expectation in equilibrium, which we will make clear later. This specification of investor belief is motivated by a discretization of the more flexible Beta distribution: $p_{t+1} \sim L \cdot \text{Beta}(\alpha_t, 1/\alpha_t)$, and $\alpha_t > 0$ parameterizes the density on higher values of

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5It is not difficult to generalize our model to $r > 0$ by introducing a growth rate on the population of agents or/and the dollar holding of each agent. The qualitative results will remain the same, albeit with more complex algebra.
The single parameter $\omega_t$ also captures the level of investor confidence in the coin’s price next period. A larger value of $\omega_t$ simultaneously implies a stronger belief that a crash will not happen tomorrow and a higher expected return tomorrow: $\mathbb{E}(p_{t+1}) = L\omega_t^2$. The notion of investor confidence is borrowed from Weil (1987), who shows how currency bubbles can emerge in an overlapping generations (OLG) model similar to ours.\footnote{In the appendix, we show the one-to-one mapping between $\omega_t \in (0,1)$ and $\alpha_t \in (0, +\infty)$ and employ the more general version of beliefs in numerical exercises. To obtain analytical solutions, we make use of the discrete version in the main text.}

Let $y$ be the amount of dollars that an investor holds. Specifically, an investor in period $t$ is willing to buy coins at price $p_t$ only if the expected return $\mathbb{E}(p_{t+1}) \geq 1$, or equivalently,

$$\frac{L}{p_t} \omega_t^2 \geq 1. \tag{2}$$

This describes the behavior of investors, who are price takers. Given a price $p_t$, if confidence $\omega_t$ is such that (2) holds with strict inequality, then each investor’s demand for coins is positive at $y/p_t > 0$. If the opposite of (2) holds, the demand is zero.

We suppose that there is a constant supply of coins $A$ for all $t$. Let $N > 0$ be the total mass of investors. Let $n_t < N$ be the mass of investors that buy coins in period $t$. The remaining $N - n_t$ do not buy. Then, for the market to clear, we require the condition:

$$\frac{yn_t}{p_t} = A.$$ 

The highest possible price level, $L$, happens when every investor demands coins, so

$$\frac{yN}{L} = A,$$ 

or simply $L = yN/A$. For any $p_t \in (0, L)$, we must have some investors buying coins while some others do not buy (i.e., $0 < n_t < N$).\footnote{In Weil (1987), however, the confidence is fixed constant. As a consequence, the speculative price does not “surge up” as what we often observe in real bubbles. We let confidence be time-varying to more closely capture price surges. For further details on this, see Appendix A.1.} This requires investors to be indif-

\footnote{It can be shown that if $p_t = 0$ (or $p_t = L$) for any period $t$, then the price must be zero (or $L$, respectively).}
ferent between buying the coins and buying the outside good; (2) holds with equality: \( L\omega_t^2/p_t = 1 \). This bears the same spirit as the “arbitrage-free condition” that underlies the efficient market hypothesis in finance\(^9\).

Throughout this paper, we employ the notion of rational expectations to compare outcomes across various settings. This requires that investors are correct about tomorrow’s price in equilibrium. In other words, prices are realized exactly according to investors’ beliefs in equilibrium. So as long as the bubble is sustained (i.e., a crash has not happened), we have

\[
p_{t+1} = L\omega_t,
\]

which, together with equality in (2), implies the following recursive relationship

\[
\omega_{t+1} = \sqrt{\omega_t}.
\]

This equation characterizes the evolution of investor’s confidence in equilibrium. We shall focus on interior equilibria in which confidence is neither 0 or 1, and therefore a crash remains a meaningful, but uncertain, possibility. What we have characterized so far, a bubbly equilibrium, signifies a realized event in which investors’ demand sustains a positive price from period to period. The price path is supported by demand from investors, who rationally expect such price path as well as the risk of crash. We formally define a bubbly equilibrium as follows.

**Definition 1.** Given \( A, y, \) and \( N \), an bubbly equilibrium is a sequence \( \{\omega_t, p_t, n_t\} \) for \( t = 1, 2, ..., T \) such that \( p_t > 0, \forall t \) and: (i) market clears in every period: \( y n_t/p_t = A \), (ii) the expected return is equal to 1 in every period: \( L\omega_t^2/p_t = 1 \), and (iii) price \( p_{t+1} \) is rationally expected as \( \omega_t L \) in every period.

It is important to note that a bubbly equilibrium is defined conditional on the bubble being sustained. When we say that \( \{p_t\} \) is part of a bubbly equilibrium, we mean that these will be the realized prices as long as the bubble does not crash. It is understood that in every period \( t \), there is probability \( 1 - \omega_{t-1} \) that the price ends up respectively) for all periods.

\(^9\)The best known application of this condition is perhaps the Arbitrage Pricing Theory, introduced by Roll and Ross (1976). Also see Roll and Ross (1980) and a more recent discussion in Malkiel (2003).
being 0 (the bubble bursts). When such an event happens, the prices and investor sizes will be zero thereafter.

We have shown that a bubbly equilibrium uniquely exists for any given value of the initial \( \omega_1 \). Specifically, \( \omega_2 = \sqrt{\omega_1}, \omega_3 = \sqrt{\omega_2}, \ldots, p_1 = L\omega^2_1, p_2 = L\omega^2_2, \ldots, \) and \( n_1 = Ap_1/y, n_2 = Ap_2/y, \ldots \) This establishes that a speculative bubble can be raised on an otherwise value-less item, where both the prices and the crash of the bubble are essentially self-fulfilled beliefs of investors – completely detached from the fundamental value of the item (zero). Later, in this section, we show how investor’s demand for currency can be augmented with fundamental value beyond pure speculation.

The sequence of equilibrium beliefs \( \{\omega_t\} \) implied by Equation (3) can be completely determined by investors’ initial beliefs, or confidence level, \( \omega_1 \). For any confidence level \( \omega_1 \in (0, 1) \), Equation (3) implies an increasing sequence of equilibrium beliefs in which \( \omega_t < \omega_{t+1} \). Further, this requires a sequence of prices \( \{p_t\} \) that also increases toward \( L \) over time. Thus we have the following mirror intuition: higher and higher confidence is required to sustain investors’ demand in face of increasing price, while simultaneously, the increasing price fulfills the higher and higher beliefs over time. This is the “bubble” environment we wish to capture in the extended model with user adoption.

Finally, to ensure any bubbly equilibrium does not unravel, we must specify two boundary conditions. First, we must assume that the investors in period \( T + 1 \) are guaranteed an “exit interest rate” equal to 1. This assumption effectively eliminates the “end-period effect” because the investors in last period need not form a belief on tomorrow’s price. So \( \omega_t \) increases as \( t \) increases until \( t = T \). As a result, here \( \omega_T \) is the most demanding belief in the sequence \( \{\omega_t\}_{t=1}^T \).

The second boundary condition regards bounds on initial beliefs, \( \omega_1 \). Suppose that we know a priori that the confidence among investors is bounded above by \( \bar{\omega} < 1 \), or in other words, the most demanding level of confidence that investors can sustain a bubbly equilibrium. We can think of \( \bar{\omega} \) as a primitive property of investors in the market. For a bubbly path to sustain, therefore, we would require \( \omega_1 \) to be sufficiently small such that \( \omega_T \leq \bar{\omega}, \) or simply that \( \omega_1 \leq (\bar{\omega})^{1/T}. \) Otherwise, the entire bubbly path will unravel from some period and no bubbly equilibrium can exist. For any \( \bar{\omega} > 0, \) one can always use a sufficiently small \( \omega_1 \) such that \( \omega_T \leq \bar{\omega}, \) so that a bubbly path exists. However, smaller values of \( \omega_1 \) increase the likelihood the bubble is to crash at some \( t < T. \) So,
even though we can always find feasible bubbly equilibrium for any given $\omega$, a smaller $\bar{\omega}$ makes it hard for the bubble to either form or sustain.

### 2.2 User Demand

Next, we consider a mass of $M = 1$ potential users who desire coins purely as a medium of exchange. For exposition, we first consider the case where no investors are present. Later, we will consider the case where both investors and users are present. We employ a Bass-model-like adoption. Suppose there are $m_t$ users who have already adopted the currency in a given period $t$. Then a user $i$ has the following utility for adoption in period $t+1$:

$$U_{i,t+1} = V(m_t) - c_i$$

where $c_i \geq 0$ is the user’s cost and $V(m_t)$ is the benefit from adoption. The user’s benefit from adoption is the ability to conduct transactions, which is increasing in the number of other users. This feature captures the network effects inherent in currency markets. For simplicity, we assume the specification $V(m_t) = m_t$.

This adoption cost $c_i$ represents the user’s opportunity cost of using the currency. In the case of a cryptocurrency, it can be thought of the time and effort to maintain the required softwares, as well as privacy risks. We let this cost be distributed among the $M = 1$ users according to a c.d.f. $G(\cdot)$. To obtain analytical solutions, we will make a linear specification:

$$G(c) = \begin{cases} 
\delta + \lambda c, & \text{if } 0 \leq c < \frac{1-\delta}{\lambda}; \\
1, & \text{otherwise},
\end{cases}$$

where $\delta$ is the set of consumers with zero adoption costs and $\lambda > 0$ captures the skewness toward low adoption costs. For instance, a higher value of $\lambda$ means that users

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10. $M = 1$ is a normalization, which means that $N$ is interpreted as the size of investors relative to the size of users.

11. Contrary to some misconceptions, using cryptocurrency bears considerable privacy risks. For one thing, transactions of block-chain based currencies are entirely public, recorded in an open ledger that anyone can query at any time. Such openness underlies the operations of blockchain. A metaphor is that, instead of using a secret vault, blockchain ensures the integrity of a document by displaying it in the town square under the public eye.
have uniformly lower adoption costs. In a given period $t$, a user $i$ adopts the currency if and only if $V(m_{t-1}) > c_i$. For the first period, $m_1 = \delta$ naturally represents the the size of the initial installed base of users. This adoption rule implies that the adoption dynamics, represented by the sequence $\{m_t\}_{t=1}^T$, satisfies

$$m_{t+1} = \delta + \lambda m_t.$$  

(5)

Under the assumed distribution on adoption costs in (4) and the linearity of network effects $V(m_t) = m_t$, the adoption path $\{m_t\}$ will converge to some value $m^* \in (0, M)$. This convergent point is defined by the identity $m^* = \delta + \lambda m^*$, or equivalently

$$m^* = \frac{\delta}{1 - \lambda},$$  

(6)

which is well-defined as long as $\lambda < 1$. We refer to $m^*$ as the **natural adoption ceiling** as it refers the level of adoption that evolves without the influence of investors.

Let $p_t > 0$ be the price of a coin in period $t$ and suppose there is a total of $A$ coins in existence. If each user demands $x > 0$ dollars worth of coins to do exchanges then the total demand for coins is given by $m_t x / p_t$. In order to match supply and demand for coins, we impose the **market-clearing condition** for users only:

$$\frac{x \cdot m_t}{p_t} = A.$$  

Equation (5) and the marketing clearing condition above imply a sequence of prices and adoption levels:

$$m_t = \frac{1 - \lambda^{t+1}}{1 - \lambda} \cdot \delta,$$  

(7)

$$p_t = \frac{x}{A} \left( \frac{1 - \lambda^{t+1}}{1 - \lambda} \right) \cdot \delta.$$  

(8)

Adoption levels increase over time, which pushes up the coin’s price concurrently. Increases in either the relative adoption benefit, as measured by $\lambda$, or the initial user base, as measured by $\delta$, raise adoption and, correspondingly, prices in every period. Finally, an increase in the supply $A$ of currency has no affect on adoption and the real value
of coins. Rather, it only pushes down the nominal value of coins so that the market clears.

2.3 Equilibrium with Users and Investors

We now describe our main model, which brings together the two isolated settings described above. They are tied together by the fact that investors and adopters pay the same price for the currency. Furthermore, investors are also adopters and use the currency as payments. Therefore, a user’s benefit now depends on the number of both users and investors:

\[ V(m_t + n_t) = m_t + n_t. \]

In period \( t + 1 \), a user with adoption cost \( c_i \) adopts the currency if \( V(m_t + n_t) > c_i \), where \( c_i \) is distributed among users according to (4). We will require \( \lambda < \frac{1-\delta}{1+N} \) so that the support of \( G \) covers \([0, M+N]\). Suppose that the number of investors active in period \( t \) is \( n_t \), then we have the following adoption dynamics (compared to (5)):

\[ m_{t+1} = G(m_t + n_t) = \delta + \lambda(m_t + n_t). \] (9)

Evident from these dynamics is the role of investors in accelerating adoption due to the currency’s inherent network externality.

Now, consider an investor’s decision. As before, an investor will purchase coins when he expects a non-negative return. However, different from before, the adoption dynamics of users implies that there may be a positive non-speculative value of the currency. Particularly, in the event of a crash, the price will drop to not zero, but rather this positive value supported by the presence of users. We call it the currency’s fundamental value. It represents the value of the coin coming purely from users’ demand for transactions. A bubble manifests itself as a detached of price from the fundamental value. In light of this feature, we augment an investor’s belief from (1) as follows.

\[ p_{t+1} = \begin{cases} \omega_t L_t + S_t, & \text{with probability } \omega_t; \\ S_t, & \text{with probability } 1 - \omega_t. \end{cases} \] (10)
In the above, $S_t > 0$ denotes the fundamental value expected in $t$ for period $t+1$. Under rational expectation, it equals the price in $t+1$ that can be supported even without any investor buying the currency ($n_{t+1} = 0$). At the other extreme, $L_t$ denotes the maximum price detachment from $S_t$. Formally, $L_t + S_t$ will be the price in $t+1$ if all investors decide to buy the currency ($n_{t+1} = N$). Later, we derive expressions for $S_t$ and $L_t$ under the rational expectation condition in equilibrium. By decomposing the price in this way, we can isolate the impacts of user adoption and investor speculation on the prices in a bubbly equilibrium.

In the investor-only model (Section 2.1), investing in the fundamental value (constant at zero) of the currency gives a return of zero. This no longer holds here because adoption of users suggests a growth in the fundamental value. The speculative investors, as a result, are more willing to participate. From a modeling perspective, this gives a rather obvious advantage for the setting with users to give rise to bubbles. From a practical perspective, we also find it unrealistic to assume a definitive user growth in a new currency (especially cryptocurrency) without any risk of disruption. Such risk can be represented by events where currency are found to be technologically flawed or removed by government decree, both of which actually happened in the realm of blockchain-powered currencies.\footnote{As to the technological risk, an example is the July 2016 attack on DAO, a blockchain-based platform, which stole $50$ million worth of cryptocurrencies. As to legal risk, an example is the China ban on bitcoin trading in 2017.}

To these ends, we introduce a sequence $\{\beta_t\}$, where $\beta_t \in [0, 1]$ denotes the survival probability of the currency to period $t+1$. In other words, $1 - \beta_t$ is what investors in period $t$ rationally expect as the probability that the currency will cease to exist tomorrow. From the modeling perspective, the goal of $\beta$ is that we can compare the model with and without users on an equal ground. To do so, we specify the weakest form of the survival probability so as to keep the return from investing in the fundamental value equal to the return of the outside option (zero). This is achieved by considering the return in period $t$ if no speculative investors participate in the market. Formally, today’s price, as prescribed by the market clearing condition, is $p_t = \frac{r m_t}{A}$. Then tomorrow’s price will be $p_{t+1} = \frac{r m_{t+1}}{A}$ where $m_{t+1}$ is given by $\delta + \lambda m_t$. So taking the survival
probability into account, the return is

$$\beta_t \frac{p_{t+1}}{p_t} = \beta_t \left( \frac{\delta}{m_t} + \lambda \right).$$

Notice that this expression does not account for the risk of a crash because the proposed investment has already precluded speculative investors. Setting the above to 1 specifies the sequence of survival probabilities:

$$\beta_t = 1 + \lambda + \frac{\delta}{m_t}.$$

It is seen that $\beta_t$ gradually increases towards 1 as $m_t$ moves to $m^*$, which we interpret as the survival probability improving as the adoption matures. There is a caveat, however, as adoption may increase beyond the natural adoption ceiling, $m^*$, when both investors and users are present (as we will show in Section 3). Therefore, we need to cap the survival probability of the currency technology at 1:

$$\beta_t = \min \left\{ 1, \frac{1}{\lambda + \delta/m_t} \right\}. \quad (11)$$

We are now in a position to specify the demand for the currency and a formalization of equilibrium for our main model. As before, we suppose each of the $n_t$ investors seeks speculative returns by spending $y$ dollars on coins, and each of the $m_t$ adopters demands $x > 0$ dollars worth of coins to do exchanges. As a result, the total demand for coins in period $t$ is $m_t x/p_t + n_t y/p_t$.

**Definition 2.** Given a set of the model primitives $\{A, x, y, \lambda, \delta, \omega_1, N, T\}$, a sequence $\Omega \equiv \{p_t, \omega_t, m_t, n_t\}_{t=1}^T$ constitutes a *bubbly equilibrium* if it satisfies the following conditions in each period.

(i) **Market clearing.** The total demand for coins by investors and adopters equals the supply for each $t$:

$$\frac{n_t \cdot y + m_t \cdot x}{p_t} = A. \quad (12)$$

(ii) **Rational expectation.** Beliefs about the next period’s non-crash price are
exactly realized:

\[ p_{t+1} = L_t \omega_t + S_t, \forall t. \]

(iii) **Arbitrage free.** The expected return of investing in coins equals the outside good’s return (normalized to 1):

\[
\frac{\beta_t \mathbb{E}(p_{t+1})}{p_t} = \frac{\beta_t}{p_t} (L_t \omega_t^2 + S_t) = 1.
\]

Applying the equilibrium definition, we see that the lowest possible price expected for period \( t+1 \), denoted by \( S_t \), must satisfy market clearing with no investors \( (n_{t+1} = 0) \):

\[
\frac{x \cdot m_{t+1} + 0}{S_t} = A \Rightarrow S_t = \frac{x}{A} \cdot G(m_t).
\]  

(13)

The currency’s fundamental value, \( S_t \), is increasing in the user demand \( x/A \) and closely tied to the level of adoption \( m_t \), per network effects. The currency’s maximal speculative value, \( L_t \), is derived from the maximal price \( S_t + L_t \) that clears the market with full investor demand \( (n_{t+1} = N) \):

\[
\frac{x \cdot m_{t+1} + yN}{S_t + L_t} = A \Rightarrow L_t = L \equiv \frac{Ny}{A}.
\]  

(14)

Note that \( L_t \) is time-invariant because the maximum speculative value that it depicts is sustained through the demand from exactly \( N \) investors. By contrast, \( S_t \) depends on the number of currency users \( (n_{t+1}) \), which is state-dependent. In the next section, we derive several important properties of the bubbly equilibrium, which convey our main results.

3 Properties of the Bubbly Equilibrium

In the following we will let \( A = x = 1 \), which is a normalization that does not affect the key features of our mechanism but significantly simplifies the exposition. Any change
in $A$ or a simultaneous change in both $x$ and $y$ only nominally inflate or deflate prices in all periods by a common factor without affecting any other aspect of a bubble. For instance, doubling both $x$ and $y$ simply deflates prices in all periods by a factor of 2.

With $x$ fixed at 1, the interpretation of $y$ is the investor’s impact on the currency market relative to the user’s. Intuitively, the larger $y$ is, the more relative influence each investor has on prices. However, $y$ does not directly change the influence each investor has on adoption. So by varying $y$, we decouple investors’ role on the financial side (influencing price) and their role on the adoption side. First, we study the special case of $y = 1$ in order to isolate the effect that investors’ speculative demand has on user adoption. Next, we will relax this restriction to identify the boundary conditions for this effect. In particular, we show in Section 3.2 that existence of a bubbly equilibrium with user adoption requires that investor demand not be too weak or too strong.

### 3.1 Special Case $y = 1$

The case of $y = 1$ most clearly exposes the intuition of our main effects: (i) a bubble accelerates user adoption, (ii) a bubble with users is less demanding on the investor’s confidence than an investor-only bubble, and (iii) a bubble with users present is less likely to crash than an investor-only bubble. The derivations of Eq. (15)-(17) below are all provided in the appendix under a more general condition.

For ease of exposition here, denote $\tilde{N} \equiv \frac{\lambda N}{\delta}$. The unique bubbly path satisfies:

$$m_{t+1} = \begin{cases} 
\delta + \lambda m_t + \tilde{N} \omega^2 m_t, & \text{if } m_t < m^*; \\
m^* \cdot \left(1 + \tilde{N} \omega^2\right), & \text{if } m_t \geq m^*.
\end{cases}$$

(15)

Later we will show that $m_1, m_2, ...$ is always a strictly increasing sequence (bounded above by 1). Observe that adoption with investors eventually exceeds the natural adoption ceiling, $m^*$. It is seen that for the periods where $m_t < m^*$, the additional term

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13One can interpret $y$ as each investor’s holdings in a base currency, such as dollars, that she uses to buy the new currency. Essentially, in our model the investor’s dollar holding calibrates the relative influence that an investor has on prices compared to her influence on adoption. One can, alternatively, set $y = 1$ and calibrate the model with $x \neq 1$ to achieve the same conditions.
Notes: This plot shows that a bubble, if sustained, entails faster adoptions. For the top curve, total investors \( N = 0.5 \). For the middle curve, \( N = 0.2 \). For the bottom curve, \( N = 0 \) which is the user-only setting. Other primitives: \( \lambda = 0.5, x = y = A = 1, \delta = 0.1, \omega_1 = 0.33. \)

Figure 2: Paths of \( \{ \omega_t \} \) under different \( \lambda \)

on the right hand side \( \tilde{N}_t \omega_t^2 m_t \) implies that \( m_t > m_t^{USE} \) (Here we use the superscript \( USE \) to indicate the bubbly path with no investors and the superscript \( INV \) for the case with no users.) The acceleration of adoption by the speculative bubble is illustrated in Figure 2. In this figure, the natural adoption ceiling is \( m^* = 0.2 \), which is the cap on adoption when \( N = 0 \). The impact of investors can be easily seen by the uniformly higher curves for \( N > 0 \). Also note the adoption curve for \( N = 0.5 \) surpasses \( m^* \) in period \( t = 3 \) while the curve for \( N = 0.1 \) surpasses \( m^* \) in period \( t = 4 \). So a larger mass of investors brings more acceleration to user adoption.

The investor beliefs satisfy:

\[
\omega_{t+1}^2 = \begin{cases} 
\frac{\delta}{m_{t+1}} \cdot \omega_t, & \text{if } m_{t+1} < m^*; \\
\omega_t - \lambda (\omega_t - \omega_t^2), & \text{if } m_t \geq m^*.
\end{cases}
\]  

(16)

Because \( m_{t+1} > m_1 = \delta \), it is seen that we have \( \omega_{t+1}^2 < \omega_t \). Comparing this to the equality \((\omega_{t+1}^{INV})^2 = \omega_t^{INV}\) that we had for the investor-only case (as Eq. (3) shows), we see that if \( \omega_1 = \omega_t^{INV} \), then in the subsequent periods \( \omega_t < \omega_t^{INV} \). In words, the presence of users lowers the requirement on investor confidence. A different interpretation here is
Notes: This plot shows that a higher $\lambda$ entails a lower requirements on confidence. For the top curve, $\lambda = 0$ which is equivalent to the investor-only case. For the middle curve, $\lambda = 0.2$. For the bottom curve, $\lambda = 0.6$. Other primitives: $N = 0.5$, $x = y = A = 1$, $\delta = 0.1$, $\omega_1 = 0.1$.

Figure 3: Paths of $\{m_t\}$ under different $N$

that if $\omega_T = \omega_T^{NV}$, then the equilibrium confidence that investors possess in the earlier periods is larger compared to the investor-only case, implying that the user-present bubble is less likely to crash.

Another property that we can see from (16) is how $\{\omega_t\}$ is affected by the model primitive $\lambda$. Recall that $\lambda$ calibrates user adoption cost: the higher $\lambda$ is, the lower adoption costs are. The pattern is clearest for the later periods where $m_t$ has surpassed $m^*$; a higher $\lambda$ implies a smaller $\omega_{t+1}$. For the earlier periods, the size of $\omega_{t+1}$ depends on $m_{t+1}$. However, it is intuitive that lower adoption costs should lead to faster adoption and a higher $m_{t+1}$. So larger values of $\lambda$ should again imply a lower $\omega_{t+1}$. Together, we see that the requirement on investor confidences is relaxed when the adoption costs are lowered among users. This is illustrated in Figure 3.

Finally, with regard to the mass of investors:

$$n_t = \begin{cases} \frac{\hat{N}}{\lambda} \omega_t^2 m_t, & \text{if } m_t < m^*; \\ \frac{\hat{N}}{\lambda} \omega_t^2 m^* - (m_t - m^*), & \text{if } m_t \geq m^*. \end{cases} \quad (17)$$

The expressions above illustrate some intuition on a “regime change” in the dynamics
of investor confidence $\omega_t$, which depends on the whether or not $m_t$ has surpassed $m^*$. Just like in the investor-only model in Section 2.1 generally there is pressure for $\omega_t$ to grow over time to sustain a bubble (Later we show this more formally). For the periods in which $m_t < m^*$, it is seen from the equation above that $\omega_t^2$ is proportional to the ratio of $n_t/m_t$. So the growth in $m_t$ serves to alleviate the pressure on $\omega_t$ to increase, which is intuitive.

In contrast, for the periods where $m_t \geq m^*$, $\omega_t^2$ becomes proportional to $(n_t + m_t - m^*)/m^*$, so the growth in $m_t$ now adds to the pressure on $\omega_t$ to increase. Intuitively, this is because if the bubble is to burst in these periods, the currency’s fundamental value will drop (as $m_t$ reverts back down to $m^*$). With investors rationally expecting the possibility of such a sharp crash, a steeper increase in their confidence must be required to sustain the bubble. Interestingly, despite this, the required confidence in the post-$m^*$ periods is still lower compared to the setting where users are absent.

### 3.2 General Case

Next, we consider the general case $y \neq 1$. First, we generalize the recursive characterizations of the bubbly path in Section 3.1.

**Lemma 1.** With $y > \lambda$, a sequence $\Omega \equiv \{p_t, \omega_t, m_t, n_t\}_{t=1}^T$ constitutes a bubbly equilibrium iff $n_t \in [0, N]$, $\omega_t \in (0, 1)$, $m_t \in [0, 1]$ for all $t$, and user adoption evolves as follows

$$m_{t+1} = \Xi(\omega_t, m_t) \equiv \begin{cases} \delta + \lambda m_t + \frac{\lambda \omega^2 N}{y - \lambda}, & \text{if } m_t < m^*; \\ \frac{\delta y + \lambda \omega^2 N y + (y-1)\lambda m_t}{y - \lambda}, & \text{if } m_t \geq m^*, \end{cases}$$

and the investor beliefs evolves as follows

$$\omega_{t+1}^2 = \Psi(\omega_t, m_t) \equiv \begin{cases} \omega_t \delta \left( \delta + \lambda m_t + \frac{\lambda \omega^2 N}{y - \lambda} \right)^{-1} + \omega_t \frac{y - 1}{y}, & \text{if } m_t < m^* \text{ and } m_{t+1} < m^*; \\ \omega_t \delta + \frac{\lambda \omega^2 N}{y - \lambda} + \frac{\lambda m_t (1 - \lambda)}{N y} - \frac{\lambda y}{N y}, & \text{if } m_t < m^* \text{ and } m_{t+1} \geq m^*; \\ \omega_t - \frac{\lambda \omega^2}{y - \lambda} + \frac{(1 - \lambda) \lambda \omega^2}{y - \lambda} + \frac{\lambda (y-1)(m_t - \lambda m_t - \delta)}{(y - \lambda) N y}, & \text{if } m_t \geq m^*. \end{cases}$$
and the mass of investors satisfies

\[ n_t = \begin{cases} 
\frac{\omega^2 N}{\lambda - \lambda/y + \delta/m_t}, & \text{if } m_t < m^*; \\
\frac{\omega^2 N y + \delta - (1 - \lambda) m_t}{y - \lambda}, & \text{if } m_t \geq m^*.
\end{cases} \]

Together, \( \Xi \) and \( \Psi \) recursively describe the entire bubbly equilibrium path. The derivations leading to these recursive equations are shown in the appendix. Lemma 1 implies the following convenient result.

**Corollary 1.** On a bubbly equilibrium, we have \( n_{t+1} = N \omega_t \) for all \( t \).

To extend the main result in Section 3.1 to \( y \neq 1 \), the first technical issue is the existence of a bubbly equilibrium. The characterization in Lemma 1 tells us that, if an equilibrium exists, then it is unique. However, (18)-(20) does not guarantee that \( n_t \in [0, N] \), \( \omega_t \in (0, 1) \) and \( m_t \in [0, 1] \) for all \( t \leq T \). If any of these conditions fail as we enter into a period, then the bubble as proposed by recursive characterization cannot be formed and no bubbly equilibrium exists. If fact, the following two examples demonstrate non-existence when \( y \) takes relatively extreme values. We lay out the general intuition behind the non-existence after the examples.

**Example 1.** (Non-existence when \( y \) is small) Let \( N = \frac{1}{2}, \delta = \frac{1}{4}, \lambda = \frac{1}{2}, \) and \( y = \frac{1}{4} \). Consider the first period. Market clearing says

\[ \frac{n_1 y + \delta}{p_1} = A = 1 \Rightarrow p_1 = n_1 y + \delta. \]

It should be immediate that a higher \( n_1 \) will lead to a higher price \( p_1 \). Next, notice that \( m_2 = \delta + \lambda(\delta + n_1) \) and the fundamental value tomorrow is expected as \( S_1 = \frac{x}{A} m_2 = m_2 \). In addition, \( L = Ny \). The key observation here is that \( S_1 \) responds to \( n_1 \) at a faster rate than \( p_1 \). What implication does this have on the expected return? The rationally expected return is

\[ \beta_1 \cdot \frac{L \omega^2 + S_1}{p_1} = \frac{4}{3} - \frac{1}{3} \cdot \frac{1 - \omega^2}{1 + n_1}. \]
This return is increasing in $n_1$. Further, at $n_1 = 0$, the return is $1 + \omega_1^2 / 3$ which is already larger than 1. Further, this return is increasing in $n_1$. As a result, the arbitrage-free condition cannot be satisfied and the financial market cannot clear. ■

Example 2. (Non-existence when $y$ is large) Consider a series of models where $Ny = 1$ but allowing $y \rightarrow +\infty$. Intuitively, this keeps the total buying power of investors constant (as well as their role on the financial side) but diminishes their role on the adoption side down to zero. As a result, the adoption rate follows the user-only case: $m_2 \rightarrow \delta + \lambda \delta$ and $m_3 \rightarrow \delta + \lambda \delta + \lambda^2 \delta$ and so on. We will focus on the second-period expected return for investors. To do that, we start with the second-period market clearing:

$$\frac{n_2 y + m_2}{p_2} = A = 1 \Rightarrow p_2 = n_2 y + m_2.$$ 

Furthermore, rational expectation requires that $p_2 = S_1 + \omega_1 L$ where $S_1 = m_2$ and $L = Ny = 1$. In addition, $S_2 = m_3$. So we have the expected return in the second period as

$$\beta_2 \cdot \frac{L \omega_2^2 + S_2}{p_2} = \frac{1}{\lambda + \delta/m_2} \cdot \frac{Ny \omega_2^2 + m_3}{N y \omega_1 + m_2} \rightarrow \frac{1 + \omega_2^2/m_3}{1 + \omega_1/m_2}.$$ 

For this expected return to be equal to 1, we must have $\omega_2^2 = \frac{m_3}{m_2} \omega_1 = \frac{\delta + \lambda \delta + \lambda^2 \delta}{\delta + \lambda \delta} \omega_1$. When $\omega_1$ is close to 1, $\omega_2$ has to be bigger than 1 which is impossible. ■

As it turns out, the existence of a bubbly equilibrium cannot be guaranteed if $y < 1$ or $y > 1 + \frac{1}{\lambda}$. Proposition 1 formalizes our existence result.\textsuperscript{14}

**Proposition 1.** (i) If $1 \leq y \leq 1 + \frac{1}{\lambda}$, then a bubbly equilibrium path (uniquely) exists for any $T$. (ii) Otherwise, there always exists sets of primitives such that no bubbly equilibrium exists.

\textsuperscript{14}The condition on $y$ can be stated in on other parameters of our model. As noted above, we focus on $y$ as convenient instrument that, when $x = 1$, calibrates the impact that investors have on the price of the currency relative to users.
Intuitively, in our model, investors have a financial role (influencing today’s price) as well as a role in adoption (influencing user adoption). When \( y < 1 \), investors have a stronger influence on the adoption side relative to the financial side (when compared to users). As more investors participate, it drives up today’s price due to speculative demand, but it also increases tomorrow’s price due to adoption acceleration. With \( y < 1 \), the adoption effect overwhelms the speculative demand effect, leading to higher expected return. Normally, for the market of a financial asset to clear, the return of that asset should fall as more investors come in to buy the asset (so that the arbitrage opportunity will close or dissipate as investors exploit it). However, the dynamics with \( y < 1 \) here raise the return instead of suppressing it. Consequently, the financial market cannot clear, as illustrated in Example 1.

By contrast, when \( y > 1 + \frac{1}{\lambda} \), investors have a greatly stronger influence on the financial side than the adoption side. As investors buy today, it significantly drives up today’s price but produces relatively little lift on tomorrow’s price. The only way to maintain the bubble is for investor’s confidence to rise faster, which makes a bubble harder to sustain. With a very large \( y \), prices are almost completely driven by investors (i.e., users have little impact on prices) and consequently are largely speculative rather than fundamental. To satiate such demand requires impossibly high confidence levels and an equilibrium cannot be achieved, as illustrated in Example 2.

Now, with the existence established, next we move to displaying several properties of the unique bubbly equilibrium as characterized by Lemma 1.

**Proposition 2.** Under the existence condition in Proposition 1, adoption by users is strictly increasing in \( t \): \( m_{t+1} > m_t \) for all \( t \) on the bubbly path.\(^{15}\)

This result says that there are additional users in every period over time. Though an immediate property in the Bass model, it is by no means easily granted in our model. It is conceivable that the presence of speculative investors could distort the adoption process so that the latter becomes non-monotonic. But Proposition 2 says that this cannot be the case. The intuition behind this unusually regular result has its roots in the assumption of rational expectation, which asserts that if there were a period in which the user mass declines, investors in the prior period would have correctly expected

\(^{15}\)The condition in this proposition can be relaxed to \( y \geq 1 \), given that a bubbly equilibrium exists.
it. To keep the currency a worthy investment, the anticipation of user decline (which translates to a decline in fundamental value) needs to be compensated by a much higher confidence level, which makes a bubble difficult to sustain. In other words, an expectation of growth in users is necessary to keep short-term investors willing to participate in the market. As that expectation is rational, the user base must actually grow in every period.

Let \( t^* = \min \{ t : m_t \geq m^* \} \) be the first period where the adoption level exceeds the natural adoption rate \( m^* \). Proposition 2 says that \( t^* \) is a clean cutoff in the sense that once \( m_t \) goes above \( m^* \), it will stay that way. However, the proposition does not provide for the existence of \( t^* \). It turns out that, as long as the primitives of the model are such that a bubbly path exists and progresses long enough, then there always will be a period at which the natural adoption ceiling \( m^* \) is surpassed.

**Proposition 3.** Under the existence condition in Proposition 1, there always exists a period \( t^* \) such that \( m_t > m^* \) for all \( t > t^* \).

The previous two results, Propositions 2 and 3, formalize our main results regarding adoption and how it is affected by the presence of investors. We now present our main results of the reverse influence.

**Proposition 4.** Under the existence condition in Proposition 1, \( \omega_{t+1}^2 < \omega_t \) for all \( t \) on the bubbly path\(^{16} \).

In words, the presence of users will relax the requirements on investor confidence. Next, suppose that users are already present, we want to move a step further by asking how the potential of adoption among users affects the requirements on investor confidence. In our model, this adoption potential is calibrated by the adoption cost parameter \( \lambda \). In addition, it can be shown that \( \{\omega_t\} \) is eventually an increasing sequence. So the most demanding requirements on investor’s confidence come in the later periods. The next proposition makes it clear that in these later periods, \( \omega_t \) is decreasing in \( \lambda \).

\(^{16}\)For a more relaxed condition \( y \geq 1 \), it can be shown that \( \omega_{t+1}^2 < \omega_t \) for all \( t > t^* \), given that a bubbly equilibrium exists.
Proposition 5. Consider \( \lambda < \lambda' \), with every other model primitive being equal. Let \( \{\omega_t\} \) be a part of the bubbly path under \( \lambda \) and \( \{\omega'_t\} \) be a part of the bubbly path under \( \lambda' \). There always exists a period \( s \) large enough such that for all \( t \geq s \), we have \( \omega'_t < \omega_t \).

Investors expecting a more steady stream of users to buy currency do not need to rely as heavily on the confidence of their fellow investors to sustain a return. This intuition is consistent with our observations from the introduction about the impact of the Shared-Coin service, which can be viewed in our model as an increase in \( \lambda \) (a reduction in adoption cost). Proposition 5 says that a currency bubble is most likely to form (or less likely to crash once formed) when some event lowers the adoption costs among potential users.

4 Conclusion

Our study is the first to look into the possible interactions between financial bubbles and product diffusion for new currency adoption. We provide insights on when and how investors and users reinforce each other in sustaining speculative returns and adoption benefits, respectively. Taking our results a step further, it seems to suggest that any entity introducing a currency can take advantage of bubbles to induce currency adoption. Indeed, it is worthwhile to note that the European Monetary Union initially launched the Euro in 1999 to investors only, before it became the EU’s standard medium of exchange. Our theory provides a rationale for such a launch strategy as it shows the synergy between users and investors. Lessons from our model may be more widely applied now that non-government entities can easily market their own currencies.

References


A Appendix

A.1 Continuous beliefs

For our analytical results, we relied on a simplified discrete system of investor beliefs (see (1)). This simplification allowed us to cleanly explicate the interplay between users and investors with analytical results. But, as Figure 1 (bitcoin prices) illustrates, real currency prices fluctuate more rapidly than abstracted by our bubbly equilibria of section 4. In this section, we show that this difference can be reconciled by applying a continuous distribution on investor beliefs. Specifically, we modify (1) to a general Beta distribution and display the equilibrium paths numerically. This allows us to illustrate how the mechanism in our simple model may appear in a “dirtier,” more realistic, market setting.

Let us consider

\[ p_{t+1} \sim L \times \text{Beta}(\alpha_t, 1/\alpha_t) \]  

As Figure 4 illustrates, the mass of this distribution moves away from 0 towards \( L \) as \( \alpha_t \) increases from 0 to +\( \infty \). The distribution family includes the uniform distribution when \( \alpha_t = 1 \). As Figure 4 shows, the Beta family that we consider is inclusive with respect to mirror beliefs: the p.d.f. of \( \text{Beta}(\alpha_t, 1/\alpha_t) \) is exactly the mirror image of the p.d.f. of \( \text{Beta}(\alpha'_t, 1/\alpha'_t) \) with \( \alpha'_t = 1/\alpha_t \).

The two-point specification in Eq. (1) is an approximate discretization of the Beta family. To see this, suppose that we want to discretize the Beta family to a two-point distribution as follows while preserving the mean as well as variance:

\[
p_{t+1} = \begin{cases} 
    h_t \cdot L, & \text{with probability } \omega_t; \\
    0, & \text{with probability } 1 - \omega_t.
\end{cases}
\]  

(21)
As we increase $\alpha$, the belief of tomorrow’s price moves its mass from 0 towards $L$.

Figure 4: P.d.f.'s of Beta($\alpha$, $1/\alpha$)

By using the formulas for the mean and variance for a beta distribution, we have

$$h_t \omega_t = \mathbb{E}(p_{t+1}) = \frac{\alpha_t}{\alpha_t + 1/\alpha_t};$$

$$h_t^2 (1 - \omega_t) \omega_t = \text{Var}(p_{t+1}) = \frac{1}{(\alpha_t + 1/\alpha_t)^2(\alpha_t + 1/\alpha_t + 1)}.$$

These give us $h_t$ and $\omega_t$ both as functions of $\alpha_t$. There is explicit expression for $\omega_t$:

$$\omega_t = \frac{\alpha^2 + \alpha^2_t + \alpha_t}{\alpha^2 + \alpha^2_t + \alpha_t + 1}.$$

It is seen that there is a one-to-one relation between $\omega_t$ and $\alpha_t$. As $\alpha_t$ increases from 0 to $+\infty$, $\omega_t$ increases from 0 to 1. Basically $\alpha_t$ represents the investor confidence in the continuous-belief model, playing the same role as $\omega_t$ in the discrete-belief model.

There is unfortunately no explicit expression for $h_t$. But we can trace the plot of $h_t$ against $\omega_t$ as we vary $\alpha_t$, and this pins down $h_t$ as a function of $\omega_t$, which is plotted in Figure 5. It is very close to the 45 degree line. So the discretized distribution (21) is closely approximated by the belief specification that we used in the main model, that is, Eq.(1).

We can solve the investor-only model using the Beta family of beliefs instead of any discretized belief system. The advantage of doing this is that it will give us a more realistic price path. The downside, however, is that the definitions of bubble and crash
Notes: $h$ is an increasing function mapping $[0, 1]$ to $[0, 1]$. Particularly, it is very close to a linear function, with which we have derived the analytical results.

Figure 5: Plot of $h(\omega_t)$

become less clear-cut, which makes the theoretical discussions less transparent.

With $p_{t+1} \sim L \times \text{Beta}(\alpha_t, 1/\alpha_t)$, the expected return in $t$ is given by

$$\frac{L}{p_t} \cdot \frac{\alpha_t}{\alpha_t + 1/\alpha_t} = 1.$$

The bubbly path evolves as follows. In each period $t$, $p_t$ is realized through $L \times \text{Beta}(\alpha_{t-1}, 1/\alpha_{t-1})$. Next, $\alpha_t$ is pinned down through the above equation. Then we move to the next period $t + 1$.

We can also add users to the picture. The belief is $p_{t+1} \sim S_t + L \times \text{Beta}(\alpha_t, 1/\alpha_t)$, where $S_t$ and $L$ are still defined through the market clearing condition as in (13) and (14). The return in period $t$ is given by

$$\frac{\beta_t}{p_t} \left( S_t + L \cdot \frac{\alpha_t}{\alpha_t + 1/\alpha_t} \right) = 1.$$

In each period, $n_t$ and $m_t$ are still determined as before through $m_t = \delta + \lambda(m_{t-1} + n_{t-1})$ and the market clearing condition. Figure 6 illustrates several typical price paths.
Notes: This figure illustrates the richness of the continuous-belief model in terms of the price paths that it can generate. The two plots are different realizations of the equilibrium path under the continuous parameterization of investor belief. Model primitives: $\lambda = 0.5$, $\delta = 0.1$, $N = 0.5$, $x = y = A = 1$, $\alpha_1 = 1$.

Figure 6: Price path examples in continuous-belief model
A.2 Proofs

Proof of Lemma 1 Here we derive recursive formulae of (18), (19) and (20). First period adoption, \( m_1 = \delta \). Now suppose \( \Sigma \) constitutes a bubble path (i.e., no crash happening). First, using market clearing condition, we get

\[ p_1 = n_1 y + m_1. \]

Setting the expected return in \( t = 1 \) equal to 1 gives us

\[ p_1 = (L\omega_1^2 + S_1) \beta_1 = \frac{1}{\lambda + \delta/m_1} \omega_1^2 Ny + \frac{\lambda n_1}{\lambda + \delta/m_1} + m_1. \]

These two equations together yield

\[ n_1 = \frac{\omega_1^2 N}{\lambda - \lambda/y + \delta/m_1}. \]

Next, by (13) and (14),

\[ S_1 = \delta + \lambda m_1 + \frac{\lambda \omega_1^2 N}{\lambda - \lambda/y + \delta/m_1}, \quad L = Ny. \]

The second-period bubble price is rationally expected as

\[ p_2 = L\omega_1 + S_1 = \delta + \lambda m_1 + Ny\omega_1 + \frac{\lambda \omega_1^2 N}{\lambda - \lambda/y + \delta/m_1}. \quad (22) \]

The second-period mass of users is

\[ m_2 = \delta + \lambda m_1 + \lambda n_1 = \delta + \lambda m_1 + \frac{\lambda \omega_1^2 N}{\lambda - \lambda/y + \delta/m_1}; \]

which gives the period \( t = 2 \) version of (18) for \( m_t < m^* \). Market clearing in period 2 gives us

\[ n_2 = \frac{p_2 - m_2}{y} = N\omega_1. \]
Next we can compute $S_2$ by \ref{13} as
\[
S_2 = \delta + \lambda m_2 + \lambda n_2 = \delta + \lambda \delta + \lambda^2 m_1 + \frac{\lambda^2 \omega^2_1 N}{\lambda - \lambda/y + \delta/m_1} + \lambda N \omega_1. \tag{23}
\]

Setting the second-period expected return to 1, we have the expression for $\omega_2$,
\[
\omega^2_2 = \frac{p_2}{L \beta_2} - \frac{S_2}{L}.
\]

In the above, $\beta_2$ is a function $m_2$. However, unlike $m_1$, it is possible for $m_2 \geq m^* = \frac{\delta}{1 - \lambda}$ so that $\beta_2 = 1$. We carry out the calculation for $\omega_2$ separately depending on whether the second-period survival rate $\beta_2$ is below or equal to 1, which will lead us to the first two cases of \ref{19} with $t = 1$. As long as $m_t < m^*$, so that $\beta_t < 1$, one can repeat the above derivations to any later period $t$ to establish the first parts $\Xi$ and $\Psi$.

If $m_t \geq m^*$, the derivations need to be modified. First, notice that $\beta_t = 1$. Second, applying (iii) of Definition 2 in period $t$ (setting the return to unity) gives us
\[
p_t = (L \omega_t^2 + S_t) \beta_t = \omega_t^2 N y + \delta + \lambda m_t + \lambda n_t,
\]
while (i) of Definition 2 (market clearing) says
\[
p_t = n_t y + m_t.
\]

So we have
\[
n_t = \frac{\omega_t^2 N y + \delta - (1 - \lambda)m_t}{y - \lambda}.
\]

Now the next-period price is rationally expected as
\[
p_{t+1} = L \omega_t + S_t = N y \omega_t + \frac{\lambda \omega_t^2 N y - (1 - y) \lambda m_t + \delta y}{y - \lambda}.
\]

The next-period mass of users is
\[
m_{t+1} = \delta + \lambda m_t + \lambda n_t = \frac{\lambda \omega_t^2 N y + \delta y - (1 - y) \lambda m_t}{y - \lambda}.
\]
Note that for $y > \lambda$ we have $m_t \geq m^* \Rightarrow m_{t+1} \geq m^*$ on the bubbly path (so that $\beta_{t+1} = 1$ too). To see this, simply plug $m_t \geq \frac{\delta}{1-\lambda}$ into the right side of the above equation.

Now, market clearing in $t+1$ implies

$$n_{t+1} = \frac{p_{t+1} - m_{t+1}}{y} = N \omega_t.$$ 

Next,

$$S_{t+1} = \delta + \lambda m_{t+1} + \lambda n_{t+1} = \delta + \frac{\lambda^2 \omega_t^2 N y - (1 - y) \lambda^2 m_t + \lambda \delta y}{y - \lambda} + \lambda N \omega_t.$$ 

Setting the next-period expected return to 1 (also noting that $\beta_{t+1} = 1$), we end up with the third case in (19). ■

**Proof of Proposition 1**

**Part (i)** We will make use of the equilibrium characterizations in Lemma 1. Specifically, we will show that if $\omega_t \in (0, 1)$ and $m_t \in [\delta, 1]$, then under the recursive characterizations of (18), (19), and (20) we must have $n_{t+1} \in (0, N)$, $\omega_{t+1} \in (0, 1)$, and $m_{t+1} \in [\delta, 1]$. Because it is required that $\omega_1 \in (0, 1)$ and $m_1 = \delta$, the proof is then completed by showing that $n_1 \in (0, N)$.

As to the mass of users, first consider $t < t^*$. The first case of $\Xi$ applies. It is easy to see that $m_{t+1} \geq \delta$. In addition, we have

$$m_{t+1} \leq \frac{\delta}{1-\lambda} + \frac{\lambda \omega_t^2 N}{1-\lambda} < 1.$$ 

In the above, the first inequality makes use of $m_t \leq m^*$ and $y \geq 1$, while the second inequality uses $\omega_t < 1$ and $N < \frac{1-\delta}{\lambda} - 1$.

Second, consider $t \geq t^*$. The second case of $\Xi$ applies. Using $m_t \geq m^*$, one can show that $m_{t+1} \geq m^* > \delta$. On the other hand, using $m_t \leq 1$, $\omega_t < 1$, and $N < \frac{1-\delta}{\lambda} - 1$, one can show that

$$m_{t+1} < 1.$$
Next we move to investor beliefs. For the first case of $\Psi$, it is easy to see that the expression for $\omega_{t+1}^2$ is positive. On the other hand, using $\omega_t > 0$ and $m_t \geq \delta$, we have

$$\Psi \leq \frac{\omega_t}{1 + \lambda} + \omega_t \lambda \left(1 - \frac{1}{y}\right).$$

Because $y \leq 1 + \frac{1}{\lambda}$, the above leads to $\Psi \leq \omega_t < 1$, implying $\omega_{t+1} < 1$.

For the second case of $\Psi$, using $m_t < m^*$, we have

$$\Psi \leq \omega_t \left(1 - \frac{\lambda}{y}\right) + \frac{\lambda \omega_t^2 (1 - \lambda)}{y - \lambda} \leq \omega_t - \frac{\lambda}{y} \omega_t + \frac{\lambda}{y} \omega_t^2 < \omega_t < 1.$$  

On the other hand, with $m_{t+1} \geq m^*$, the first case of $\Xi$ translates into:

$$\frac{\lambda \omega_t^2 (1 - \lambda)}{\lambda y - \lambda + \delta y / m_t} \geq \frac{\delta \lambda}{Ny} - \frac{\lambda (1 - \lambda) m_t}{Ny},$$

which tells that the expression for $\omega_{t+1}^2$ in the second case of $\Psi$ is positive.

For the third case of $\Psi$, it is easy to see that the expression for $\omega_{t+1}^2$ is positive. On the other hand, we need to show that $\Psi < 1$. To show this, we use a result saying that $m_t$ is monotonically increasing over $t$. This result will be later proved in Proposition 2 under the same condition of $y$ as in the current proposition. So $m_{t+1} > m_t$, which implies $m_t > \frac{\delta + \lambda \omega_t^2 N}{1 - \lambda}$. Plugging this last inequality into the third case of $\Psi$, we have

$$\omega_{t+1}^2 \leq \omega_t + \frac{\lambda}{y} \left(-\omega_t + \omega_t^2\right) < 1.$$  

So far we have shown that if $\omega_t \in (0, 1)$ and $m_t \in [\delta, 1]$, then the same applies to $\omega_{t+1}$ and $m_{t+1}$. We also have $n_{t+1} \in (0, N)$ by simply appealing to Corollary 1. Lastly, we only need to show that $n_1$ is feasible. Use the first case in (20):

$$n_1 = \frac{\omega_1^2 N}{\lambda - \lambda/y + 1}.$$  

It is easy to see that $n_1 \in (0, N)$ as long as $y \geq 1$ and $\omega_1 \in (0, 1)$.
Part (ii) First, notice that as $m_1 = \delta$, we have

$$n_1 = \frac{\omega_1^2 N}{1 + \lambda - \lambda/y}.$$  

It is seen that if $\frac{1}{1+\lambda} \leq y < 1$, we can always set $\omega_1$ large enough such that $n_1 > N$. For $y < \frac{1}{1+\lambda}$, $n_1$ has to be negative.

Next consider any $y > 2$. Set $\delta = \frac{y^2 - 2}{y - 1}$, $\lambda = 1 - 2\delta$, and $N = \frac{\delta}{\lambda} - \delta$ (note that these satisfy the requirement that $\lambda \leq \frac{1-\delta}{1+N}$ to make the support of $G$ cover $[0, 1+N]$). Then $m^* = \frac{1}{2}$ and $m_2 \leq \delta + \lambda m_1 + \lambda N = 2\delta$. Since $y > 2 \Rightarrow \delta < \frac{1}{8}$, we have $m_2 < m^*$. Consequently we apply the first case of $\Psi$ to get

$$\omega_2^2 = \frac{\omega_1 \delta}{m_2} + \omega_1 \lambda \frac{y - 1}{y} \geq \omega_1 \left( \frac{1}{2} + \lambda \frac{y - 1}{y} \right) = \omega_1 \left[ 1 + \frac{1}{4} (1 - 2/y) \right].$$

It is clear that for $\omega_1$ large enough, $\omega_2^2 > 1$ meaning that the bubble will not work. So we have shown that for any given $y > 2$, we can find a set of primitives such that there is no bubbly equilibrium. It can be shown that the proposed set of primitives satisfies $y > 1 + \frac{1}{\lambda}$. ■

**Proof of Corollary 1** This follows from substituting (18) and (19) into (20). ■

**Proof of Proposition 2** First, suppose $m_t < m^*$. Because $n_t \geq 0$, we must have $m_{t+1} \geq \delta + \lambda m_t$ by (9). So

$$m_{t+1} - m_t \geq \delta - (1 - \lambda)m_t > 0,$$

where the last inequality comes from $m_t < m^* = \frac{\delta}{1-\lambda}$.
Next suppose \( m_t \geq m^* \). Equation (18) tells us that as long as \( y > \lambda \),
\[
m_t \geq m_{t+1} \iff m_t \geq \frac{\delta + \lambda \omega_t^2 N}{1 - \lambda}.
\]

Now suppose that \( m_t \) is not strictly increasing for \( m_t > m^* \). Then for some \( t \), \( m_{t+1} \geq m_{t+2} \) but \( m_{t+1} \geq m_t \). This implies \( m_{t+1} \geq \frac{\delta + \lambda \omega_{t+1}^2 N}{1 - \lambda} \), which by (20) implies that \( n_{t+1} \leq \omega_{t+1}^2 N \). By Corollary 1, \( n_{t+1} = \omega_t N \) so that
\[
\omega_t \leq \omega_{t+1}.
\]

By the choice of \( t \), we have \( m_t < m_{t+1} \). But (24) says that we must have \( m_t < \frac{\delta + \lambda \omega_t^2 N}{1 - \lambda} \).

So
\[
m_{t+1} < \frac{\delta y + \lambda \omega_t^2 Ny + (y - 1)\lambda \frac{\delta + \lambda \omega_t^2 N}{1 - \lambda}}{y - \lambda} = \frac{\delta + \lambda \omega_t^2 N}{1 - \lambda} \leq \frac{\delta + \lambda \omega_{t+1}^2 N}{1 - \lambda}.
\]

But this implies \( m_{t+2} > m_{t+1} \), leading to a contradiction.

If \( m_t < m^* \) and \( m_{t+1} > m_{t+2} \), then
\[
m_{t+1} < \delta + \lambda m^* + \frac{\lambda \omega_t^2 N}{\lambda - \lambda/y + \delta/m^*} = \frac{\delta}{1 - \lambda} + \frac{y \lambda \omega_t^2 N}{y - \lambda} \leq \frac{\delta + \lambda \omega_t^2 N}{1 - \lambda} \leq \frac{\delta + \lambda \omega_{t+1}^2 N}{1 - \lambda}.
\]

But by (24) this implies \( m_{t+2} > m_{t+1} \), again leading to a contradiction. Hence, the sequence \( \{m_t\} \) is strictly increasing. \( \blacksquare \)

Proof of Proposition 3. To establish the existence of \( t^* \), it is sufficient to show that the sequence \( \{m_t\} \) converges to some point exceeding \( m^* \). Because of Proposition 2, such a point exists, which we denote as \( \bar{m} \). By (18), \( \omega_t \) must converge too. Let its limit be \( \bar{\omega} \). By way of contradiction, suppose \( \bar{m} \leq m^* \). Because \( m_t \) must converge from below, by (18) we have
\[
\delta + \lambda \bar{m} + \frac{\lambda \bar{\omega}^2 N}{\lambda - \lambda/y + \delta/\bar{m}} = \bar{m}.
\]
Together with $\bar{m} \leq \frac{\delta}{1-\lambda}$, this implies that $\bar{\omega} = 0$ and $\bar{m} = \frac{\delta}{1-\lambda}$. Again because $m_t$ converges from below, the first case of $\Psi$ applies. That case implies

$$\frac{\omega_{t+1}^2}{\omega_t} \to \frac{\delta}{\delta + \lambda \bar{m}} + \frac{\lambda y - 1}{y} \geq 1 - \frac{\lambda}{y}.$$  

Because $y > \lambda$, the right hand side above is a positive constant. As a result, when the sequence $\omega_t$ gets sufficiently close to zero from above, we have $\omega_{t+1} > \omega_t$ - a contradiction to $\bar{\omega} = 0$. So we conclude that $\bar{m} > m^\ast$. ■

Proof of Proposition 4. First, for $t < t^\ast - 1$, from (19), we have

$$\omega_{t+1}^2 = \frac{\omega_t \delta}{m_{t+1}} + \omega_t \lambda \frac{y - 1}{y} \leq \omega_t \frac{1}{1 + \lambda} + \omega_t \lambda \left(1 - \frac{1}{y}\right) \leq \omega_t.$$  

In the above, the second line uses $m_{t+1} \geq m_2 \geq \delta + \lambda \delta$ and the last line uses the condition (from Lemma 1) $y \leq 1 + \frac{1}{\lambda}$.

Next, for $t = t^\ast - 1$, by (19) we have

$$\omega_{t+1}^2 = \omega_t \left(1 - \frac{\lambda}{y}\right) + \frac{\lambda \omega_t^2 (1 - \lambda)}{\lambda y - \lambda + \delta y/m_t} + \frac{\lambda m_t (1 - \lambda)}{Ny} - \frac{\lambda \delta}{Ny} < \omega_t + \left[\frac{\lambda}{\omega_t} + \frac{\lambda (1 - \lambda)}{y - \lambda} \omega_t^2\right],$$

where the inequality comes from $m_t < m^\ast = \frac{\delta}{1-\lambda}$. Notice that the term in the above square brackets must be smaller than 0, because $y \geq 1 \Rightarrow 0 < \frac{\lambda (1 - \lambda)}{y - \lambda} \leq \frac{\lambda}{y}$. Hence, we have $\omega_{t+1}^2 < \omega_t$.
Finally, for \( t > t^* - 1 \) (or equivalently \( t \geq t^* \)), from Proposition 2 and (18) we have

\[
m_{t+1} = \frac{\delta y + \lambda \omega_t^2 Ny - (1 - y) \lambda m_t}{y - \lambda} \geq m_t,
\]

which implies that

\[
m_t \leq \frac{\delta + \lambda \omega_t^2 N}{1 - \lambda}.
\]

Now plugging the RHS into (19) for \( m_t \), we have

\[
\omega_{t+1}^2 \leq \omega_t + \left[ -\frac{\lambda \omega_t}{y} + \frac{\lambda \omega_t^2}{y} \right].
\]

Notice that, because \( \omega_t < 1 \), the term in the above square brackets must be smaller than 0. So we have \( \omega_{t+1}^2 < \omega_t \) for all \( t \). ■

**Proof of Proposition 5.** We start by considering the sequence defined by

\[
\theta_t \equiv \frac{1 - \omega_{t+1}}{1 - \omega_t} > 0
\]

for an equilibrium path of \( \{\omega_t\} \) and showing that its limit exists and is increasing in \( \lambda \). From Proposition 3 we immediately know that \( m_t \to \bar{m} > m^* \). So for large enough \( t \), it must be the second case of \( \Xi \) that applies. We have

\[
\frac{\delta y + \lambda \omega^2 N y + (y - 1) \lambda \bar{m}}{y - \lambda} = \bar{m},
\]

which gives us \( \bar{m} = \frac{\delta + \lambda \omega^2 N}{1 - \lambda} \). Plugging this into the last case of \( \Psi \), we end up with

\[
-\bar{\omega} + \bar{\omega}^2 = 0,
\]

which has two solutions, 0 and 1. For the solution \( \bar{\omega} = 0 \), (25) leads to \( \bar{m} = m^* \), which contradicts Proposition 3. Hence, it must be that \( \bar{\omega} = 1 \) and \( \bar{m} = \frac{\delta + \lambda N}{1 - \lambda} \).
Consider any \( t > t^* \). By (20) and Corollary 1, we have
\[
N\omega_{t-1} = \frac{\omega_t^2 Ny + \delta - (1 - \lambda)m_t}{y - \lambda}.
\]
Using this expression to substitute \( m_t \) and \( m_{t+1} \) in (18) gives us
\[
\omega_{t+1}^2 = \lambda\omega_t^2 - \left( \lambda - \frac{\lambda}{y} \right)\omega_{t-1} + \left( 1 - \frac{\lambda}{y} \right)\omega_t,
\]
which implies
\[
(1 - \omega_{t+1}^2) = \lambda(1 - \omega_t^2) + (1 - \lambda/y)(1 - \omega_t) - (\lambda - \lambda/y)(1 - \omega_{t-1}).
\]
Dividing both sides of the above equation by \( 1 - \omega_t \), we have
\[
\theta_t(1 + \omega_{t+1}) = \lambda(1 + \omega_t) + (1 - \lambda/y) - \frac{(\lambda - \lambda/y)}{\theta_{t-1}}.
\]
Proposition 4 implies that \( (1 - \omega_{t+1})(1 + \omega_{t+1}) > 1 - \omega_t \) so as \( t \to \infty \) we have \( \theta_t \geq \frac{1}{2} \). Taking upper limit of both sides of (26) gives
\[
2 \times \limsup \theta_t = 2\lambda + 1 - \frac{\lambda}{y} - \frac{\lambda - \lambda/y}{\limsup \theta_t},
\]
which has the solution
\[
\limsup \theta_t = \frac{\left(2\lambda + 1 - \frac{\lambda}{y}\right) \pm \sqrt{\left(2\lambda + 1 - \frac{\lambda}{y}\right)^2 - 8\left(\lambda - \frac{\lambda}{y}\right)}}{4}.
\]
Note the the smaller solution is smaller than \( \frac{1}{2} \), so we must take the larger solution.
We can obtain exactly the same solution for \( \liminf \theta_t \). Hence,
\[
\lim_{t \to \infty} \theta_t = \frac{\left(2\lambda + 1 - \frac{\lambda}{y}\right) + \sqrt{\left(2\lambda + 1 - \frac{\lambda}{y}\right)^2 - 8\left(\lambda - \frac{\lambda}{y}\right)}}{4}.
\]
It can be shown that the right hand side is increasing $\lambda$. To complete the proof, consider the product

$$\prod_{s=1}^{t-1} \theta_s = \frac{1 - \omega_t}{1 - \omega_1}.$$ 

For $\lambda' > \lambda$, we have the corresponding sequence of confidence levels $\omega'_t$ and $\omega_t$, which define $\theta'_t$ and $\theta_t$. Furthermore, as just established, $\lim \theta'_t > \lim \theta_t$, which guarantees

$$\prod_{s=1}^{t-1} \frac{\theta_s}{\theta'_s} = \frac{1 - \omega_t}{1 - \omega'_t} \to 0.$$ 

Hence, there exists a period $s$ for which $1 - \omega'_t > 1 - \omega_t$ for all $t > s$. ■